Cellular Automaton Growth on $\mathbb{Z}^2$: Theorems, Examples, and Problems

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Exactly 1 Solidification

We will study the evolution starting from a single occupied cell at the origin in considerable detail. A key initial observation is that the locations $(\pm t, \pm t)$ join the crystal at time $t$: the occupied set grows along the diagonals at the fastest possible speed, $c = \sqrt{2}$. This follows from the fact that, as in any Moore neighborhood CA started from $\{0\}$, $A_t \subset B_t = \{||x||_\infty \leq t\}$, so by induction, at time $t + 1$ each corner cell of $B_{t+1} \setminus B_t$ sees only the diagonal cell that was added at time $t$. In general, by a ladder we mean any local CA configuration which propagates over time in some direction $u$, periodically in space. A $c$-ladder is a ladder which propagates with the fastest velocity allowed by $N$ (the “speed of light”). In the present case, the four diagonal trails are $c$-ladders one cell wide and with spatial period one. We will encounter more elaborate ladders later in the paper. For now, note that the Exactly 1 rule grows persistently from any finite $A_0$ since there must be extremal occupied cells in the diagonal directions which give rise to permanent $c$-ladders.

![Fig. 6. The Exactly 1 Solidification Rule, started from a singleton, after 55 updates](image)

Starting from $\{0\}$, the intricate growth of $A_t$ off the diagonals is shown in Fig. 6 at $t = 55$. One immediately notices the recurring lace-like motifs, in striking contrast to any of the crystals discussed so far. More careful scrutiny reveals an exactly recursive structure along dyadic sequences $t_n = 2^n$, which
permits us to describe the occupied set at arbitrary times in terms of the binary expansion of \( t \). This representation, in turn, lets us compute the asymptotic density with which the Exactly 1 rule fills \( \mathbb{Z}^2 \), subsequential limit shapes \( L_r \) along subsequences \( t_n = r2^n \), and boundary lengths of the \( L_r \). The same phenomenology is observed in many other intractable CA rules, so it is satisfying to quantify a “regular fractal pattern” [TM, p.39] which serves as an exactly solvable prototype for what Packard and Wolfram [PW] called growth with “corrugated boundaries.” Details follow.

Before proceeding, though, we pause to contrast the behavior of the most familiar and exactly solvable totalistic CA on \( B_1 \) which generates a fractal: \textit{xor} over the Moore neighborhood. Starting from a singleton, that model generates a space-time pattern which is a slightly more complicated three-dimensional counterpart to the Sierpinski lattice obtained from Pascal’s Triangle Modulo 2. In the same way that row \( 2^n - 1 \) of Pascal’s Triangle has all odd coefficients and row \( 2^n \) has only two, \textit{xor} on \( B_1 \) fills each quadrant of \( B_{2^n} \) with a regular array of \( 2 \times 2 \) occupied boxes surrounded by empty frames of width 1 at time \( 2^n - 1 \), and then collapses to only 9 occupied sites at time \( 2^n \). Evidently this process does not grow persistently, although it does have a disconnected limit shape \( L_r \) along each subsequence \( t_n = r2^n \) for fixed \( r \in (0, 1) \). These shapes are cross-sections through a 3D fractal; almost all have density 0. See [Wil] for an early account of the fractal structure of linear cellular automata. Linearity property (1.2 a) makes the analysis easy in comparison with Exactly 1, to which we now return.

\textit{The Exactly 1 recursion}

Let \( A_t \) be the occupied set at time \( t \) for the Exactly 1 rule started from \( A_0 = \{0\} \). At times \( t_n = 2^n \), \( n \geq 3 \), we claim that the crystal has the following properties in the octant \( \{0 \leq x_2 \leq x_1 \} \) (with analogous structure over the rest of the lattice, by symmetry):

(i) The only occupied site with \( x_1 \geq 2^n \) is \( (2^n, 2^n) \). All other sites of the form \( (2^n, x_2) \) are vacant, with at least two occupied neighbors of the form \( (2^n - 1, y) \), and so never join the crystal.

(ii) The only occupied site with \( x_2 = 0 \) is the origin, and the occupied sites with \( x_2 = 1 \) have first coordinates \( 1, 3, 7, \ldots, 2^n - 1 \).

(iii) The configuration on the lattice region bounded by \( (2^{n-1}, 2^{n-1}), (2^n - 1, 2^{n-1}) \), and \( (2^n - 1, 2^n - 1) \) is an exact translate of the configuration on the region bounded by \( (0, 0), (2^{n-1} - 1, 0) \), and \( (2^{n-1} - 1, 2^{n-1} - 1) \). The configuration on the region bounded by \( (2^{n-1}, 2^{n-1}), (2^n - 1, 2^{n-1}) \), and \( (2^n - 1, 1) \) is the mirror reflection of the same pattern. Finally, the configuration on the region bounded by \( (2^{n-1} + 1, 2), (2^n - 2, 2), \) and \( (2^{n-1} + 1, 2^{n-1} - 1) \) is an exact rotated translate of the configuration on the region bounded by \( (2, 2), (2, 2^{n-1} - 1) \), and \( (2^{n-1} - 1, 2^{n-1} - 1) \), and all sites in lattice intervals \( \{2^{n-1}\} \times [2, 2^{n-1} - 1], [2^{n-1}, 2^n - 1] \times \{0\} \) and \( [2^{n-1}, 2^n - 2] \times \{1\} \) are vacant.
Fig. 7. One octant of Exactly 1 Solidification, started from a singleton, after 8 and 32 updates.

Fig. 7 shows the configurations on the octant at time 8 \((n = 3)\) and time 32 \((n = 5)\). Properties \((i)-(iii)\) may be verified at time 8 by inspection, and at subsequent times \(t_n = 2^n\) by induction. Assuming the hypothesis for \(n\), the key observations are that \(c\)-ladders emanate from \((2^n, 2^n)\), one proceeding along the diagonal \(x_2 = x_1\), and one heading “southeast” along \(x_2 = x_1\), and that by symmetry all sites of the form \((x_1, 2^n)\) with \(2^n + 2 \leq x_2 \leq 2^n + 1\) are vacant (after time \(2^n + 1\) any such site which is unoccupied must have an even number of occupied neighbors, and so cannot join the crystal). Thus further growth within the octant is divided into three triangular regions, as in \((iii)\) but with \(n\) replaced by \(n + 1\), which evolve independently with mixed “all 1” and “all 0” boundary conditions. The first two new regions exactly replicate the conditions of the octant up to time \(2^n - 1\), and so generate the same structure as claimed. The final triangular region also replicates this structure after a slight displacement. The \(c\)-ladder along its upper edge proceeds southeast until it occupies the point \((2^{n+1} - 1, 1)\), as desired in part \((ii)\) of the induction. We omit further details, which are checked in a similar fashion.

**Evaluation of the density**

Write \(a_n = |A_{2^{n-1}}|\), and \(b_n = |A_{2^{n-1}} \cap \{x_1 \geq 0, 0 \leq x_2 \leq x_1\}|\). Note first that, dividing the lattice into octants and appealing to symmetry, \(a_n = 8(b_n - 2^n - 1) + 4(2^n - 2) + 9\) for \(n \geq 1\). Here the first term counts occupied sites not on the diagonals \(x_2 = \pm x_1\) and outside \(B_1\), the second term counts occupied sites on the diagonals and outside \(B_1\), and the third term counts occupied sites inside \(B_1\). Thus, \(a_n = 8b_n - 4 \cdot 2^n - 7\).

Next, denote the triangular regions shown in (the right half of) Fig. 7, counterclockwise from the bottom left, by I, II, III, and IV. Then the four contributions to \(b_{n+1}\) have cardinality \(b_n\) (I and IV), \(b_n - 2\) (III), and \(b_n - (2^n - 2) - n - 2\) (II). For the last formula note that, as mentioned earlier, if one cuts from I the diagonal, and the sites with \(x_2 = 1\), and the sites with \(x_2 = 0\), then the remainder is isomorphic to II.
Therefore,
\[ b_{n+1} = 4b_n - 2^n - n - 2, \]
and so
\[ a_{n+1} = 8(4b_n - 2^n - n - 2) - 4 \cdot 2^{n+1} - 7 \]
\[ = 4(8b_n - 4 \cdot 2^n - 7) + 16 \cdot 2^n + 28 - 8 \cdot 2^n - 8n - 16 - 4 \cdot 2^{n+1} - 7 \]
\[ = 4a_n - 8n + 5. \]
The solution is
\[ a_n = \frac{4^{n+2} + 24n - 7}{9}. \]
In particular we see that \( a_n/\| B_{x-1} \| \to \frac{4}{9} \), the asymptotic density, and so \( 2^{-n} A_{x^n} \to \frac{4}{9} 1_L(x)dx \), where \( L = B_1 = \{ \|x\|_\infty \leq 1 \} \), in the sense of (3.3).

Let us pause here to pose four problems. The first two are exercises for the enterprising reader. The third, motivated by empirical observation that many small seeds induce bounded perturbations of the singleton-seed recursion, may well involve substantial effort. The fourth is quite likely most difficult.

**Problem 5.** Packard and Wolfram [PW] observed similar behavior in the Exactly 1 rule on the nearest neighbor diamond (cf. rule 174, shown in their Fig. 2), although they did not identify a recursive structure or compute its asymptotic density. Mimic the derivation above to show that the density starting from a single occupied cell equals \( \frac{2}{7} \).

**Subsequential limit shapes**

Closer inspection of Exactly 1 recursive growth, driven by \( c \)-ladders, identifies asymptotic shapes \( L_r \) along all subsequences \( t_n = r2^n \) for fixed \( r \). The limits are generated by a recursive scheme reminiscent of the von Koch algorithm [vK]. To describe it, introduce the binary expansion \( r = \sum_{k_0} \infty d_k 2^{-k}, \ k_0 \ \text{an integer, and} \ d_{k_0} = 1. \) Write \( B_n = \{ x \in \mathbb{R}^2 : \|x\|_\infty \leq s \} \). Starting from \( B_{2^{-k_0}} \), successively add three translates of \( B_{2^{-k}} \) for every \( k > k_0 \) such that \( d_k = 1, \) at the “exposed” corners of each previous square. Fig. 6 is suggestive of the algorithm for \( r = .111 = \frac{7}{8} \). Note that smaller squares are added at each of four corners of the initial square, but only three of four corners are exposed at later stages. \( L_r \) is the monotone limit when this recursion is carried out for all \( k \geq k_0 \). Note that \( L_{2r} = 2L_r, \) so by normalizing suitably it suffices to consider \( r \in [\frac{1}{2}, 1) \). Now the same methods as for the case \( r = 1 \) can be used to show that \( \lim_{n \to \infty} t_n^{-1} A_{t_n} = \frac{3}{8} 1_{r \in L_r}(x)dx \) in general. Thus we have our first example where (i) of (1.5) holds, but (ii) fails, with convergence to distinct limit shapes along suitable subsequences. All of these “corrugated” subsequential limits except \( L_1 = B_1 \) are nonconvex, and as we are about to see, many (e.g., \( L_{\frac{7}{8}} \)) have genuinely fractal edges, i.e., boundary curves with self-similar pieces of dimension greater than one.
Asymptotic boundary length

Let us conclude our discussion of the Exactly 1 rule by analyzing the lengths of its asymptotic boundaries. First suppose \( r = \sum_{i=1}^{N} d_i 2^{-i} \in \left[ \frac{1}{2}, 1 \right) \), with \( d_N = 1 \), and write \( \sigma_k = \sum_{i=1}^{k} d_i \). Then it is not hard to check by induction that the boundary length \( \lambda(r) \) of \( L_r \) is given by

\[
\lambda(r) = \frac{4}{3} \left[ 1 + \frac{4}{3} \sum_{k=1}^{N} d_k 3^{\sigma_k} 2^{-k} \right].
\]

Hence, if \( r \) has a non-terminating binary expansion, 

\[
\lambda(r) = \frac{4}{3} \left[ 1 + \frac{4}{3} \sum_{k=1}^{\infty} d_k 3^{\sigma_k} 2^{-k} \right].
\]

This asymptotic boundary length \( \lambda \), as a function of \( r \), is lower semicontinuous, sometimes finite and sometimes infinite, but not continuous at any value where it is finite. Assuming that \( \sigma_k/k \to \sigma \), the Hausdorff dimension of the boundary of \( L_r \) is given by \( \max \{ 1, \sigma \ln 3 \} / \ln 2 \). Thus, the Hausdorff dimension is not continuous anywhere: within any \( \epsilon \)-neighborhood of any \( r_0 \) there is an \( L_{r_1} \) with boundary of finite length (dimension 1), an \( L_{r_2} \) with boundary of the largest possible dimension \( \ln 3 / \ln 2 \), and a limit shape with any intermediate dimension as well.

REFERENCES


